Efficient structure-preserving model reduction for nonlinear mechanical systems with application to structural dynamics

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Time-critical applications

- real-time applications
  - structural health monitoring
  - embedded control
- many-query applications
  - design optimization
  - uncertainty quantification

$$\text{inputs } \mu \rightarrow \boxed{\text{high-fidelity model}} \rightarrow \text{outputs } y$$

- **barrier**: simulation can take days on supercomputers
- model reduction

$$\text{inputs } \mu \rightarrow \boxed{\text{reduced-order model}} \rightarrow \text{outputs } y$$

  - *offline* (expensive): 'training' analyses
  - *online* (cheap): deploy low-dimensional model

Main idea

- high-fidelity model
  - *parameterized* simple mechanical system
  - nonlinear potential energy
  - Rayleigh damping
  - external force
- existing reduced-order models
  1. preserve structure, but remain expensive
  2. destroy structure, but are cheap
- **our proposed reduced-order model**
  - preserves structure and is cheap

Outline

1. Motivation
2. Problem formulation
3. Existing model-reduction techniques
   - preserves structure, but expensive
   - cheap, but destroys structure
4. Proposed method
5. Numerical example

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1 Motivation

2 Problem formulation

3 Existing model-reduction techniques
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- cheap, but destroys structure

4 Proposed method

5 Numerical example

Lagrangian description of structural dynamics

- equations of motion from finite-element discretization

$$M(\mu)\ddot{q} + C(\mu)\dot{q} + \nabla_q V(q;\mu) = f^{\mathrm{ext}}(t;\mu).$$

- can be derived via Lagrangian dynamics with five 'ingredients':
  1. configuration space $Q = \mathbb{R}^N$
  2. Riemannian metric $g(v, w; \mu) = v^T M(\mu) w$
  3. potential-energy function $V(q;\mu)$
  4. dissipation function $\mathcal{F}(\dot{q}, \mu) = \frac{1}{2}\dot{q}^T C(\mu)\dot{q}$
  5. external force derived from the Lagrange–D'Alembert principle $f^{\mathrm{ext}}(t;\mu)$
- properties 1–3 define a *simple mechanical system*
- properties 4–5 characterize *non-conservative forces*

Equations of motion: derived from five ingredients

- configuration space: $q \in Q = \mathbb{R}^N$
- kinetic energy: $T(\dot{q};\mu) = \frac{1}{2}g(\dot{q},\dot{q};\mu) = \frac{1}{2}\dot{q}^T M(\mu)\dot{q}$
- Lagrangian:

$$\begin{aligned} L(q,\dot{q};\mu) &= T(\dot{q};\mu) - V(q;\mu) \\ &= \frac{1}{2}\dot{q}^T M(\mu)\dot{q} - V(q;\mu). \end{aligned}$$

- non-conservative forces

$$F(t,q,\dot{q};\mu) = f^{\mathrm{ext}}(t;\mu) - \nabla_{\dot{q}}\mathcal{F}(\dot{q};\mu)$$

- apply forced Euler–Lagrange equations

$$\frac{d}{dt}\nabla_{\dot{q}}L(q,\dot{q};\mu) - \nabla_q L(q,\dot{q};\mu) = F(t,q,\dot{q};\mu)$$

$$\boxed{M(\mu)\ddot{q} + C(\mu)\dot{q} + \nabla_q V(q;\mu) = f^{\mathrm{ext}}(t;\mu)}$$

Key properties

- conservative mechanical systems ($F = 0$)
  - energy conservation
  - momentum conservation
  - dynamics satisfy variational principle
  - symplectic time-evolution maps
- structure-preserving time integration
  [Marsden and West, 2001, Hairer et al., 2006]
  - discrete system preserves some of the above properties
  - leads to improved long-time behavior

reduced-order models should preserve these properties

Outline

Galerkin: structure-preserving model reduction [Lall et al., 2003]

- determine low-dimensional basis $\Phi \in \mathbb{R}^{N\times m}$
  - modal decomposition, proper orthogonal decomposition
- substitute $q = \Phi q_r$ to obtain 'reduced ingredients'
  1. configuration space $Q_r = \mathbb{R}^m$ with $\mathbf{Q}_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
  2. Riemannian metric $g_r(v_r, w_r; \mu) \equiv g(\Phi v_r, \Phi w_r; \mu)$
  3. potential-energy function $V_r(q_r; \mu) \equiv V(\Phi q_r; \mu)$
  4. dissipation function $\mathcal{F}_r(\dot{q}_r; \mu) \equiv \mathcal{F}(\Phi q_r; \mu)$
  5. external force $f_r^{\mathrm{ext}} = \Phi^T f^{\mathrm{ext}}$
- forced Euler–Lagrange equations yield

$$\Phi^T M(\mu)\Phi\ddot{q}_r + \Phi^T C(\mu)\Phi\dot{q}_r + \Phi^T \nabla_q V(\Phi q_r; \mu) = \Phi^T f^{\mathrm{ext}}(t; \mu)$$

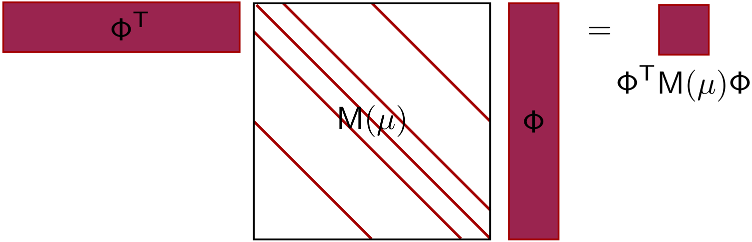
\+ preserves Lagrangian structure

\- remains expensive for parameterized, nonlinear systems

Computational bottleneck

$$\Phi^T M(\mu)\Phi\ddot{q}_r + \Phi^T C(\mu)\Phi\dot{q}_r + \Phi^T\nabla_q V(\Phi q_r;\mu) = \Phi^T f^{\mathrm{ext}}(t;\mu)$$

- when μ changes, must recompute $\Phi^T M(\mu)\Phi$ and $\Phi^T C(\mu)\Phi$
  - $\mathcal{O}(Nm^2)$ operations: scales with large dimension N



- when q_r changes, must recompute $\Phi^T\nabla_q V(\Phi q_r;\mu)$
  - $\mathcal{O}(Nm)$ operations

Outline

Collocation [Astrid et al., 2008, Ryckelynck, 2005]

$$\Phi^T M(\mu)\Phi\ddot{q}_r + \Phi^T C(\mu)\Phi\dot{q}_r + \Phi^T\nabla_q V(\Phi q_r;\mu) = \Phi^T f^{\mathrm{ext}}(t;\mu)$$

- compute subset of equations before performing Galerkin projection

  $$\Phi^T Z^T Z M(\mu)\Phi\ddot{q}_r + \Phi^T Z^T Z C(\mu)\Phi\dot{q}_r + \Phi^T Z^T Z\nabla_q V(\Phi q_r;\mu) = \Phi^T Z^T Z f^{\mathrm{ext}}(t;\mu).$$

  'sampling matrix' Z: $n_Z \ll N$ rows of identity matrix
- destroyed properties:
  2. mass matrix not symmetric: does not define a metric
  3. stiffness matrix not symmetric: does not derive from a potential-energy function
  4. dissipation matrix not symmetric: does not derive from a dissipation function

Empirical interpolation/least-squares approximation

[Grepl et al., 2007, Nguyen and Peraire, 2008, Chaturantabut et al., 2010, Carlberg et al., 2011]

$$\Phi^T M(\mu)\Phi\ddot{q}_r + \Phi^T C(\mu)\Phi\dot{q}_r + \Phi^T\nabla_q V(\Phi q_r;\mu) = \Phi^T f^{\mathrm{ext}}(t;\mu)$$

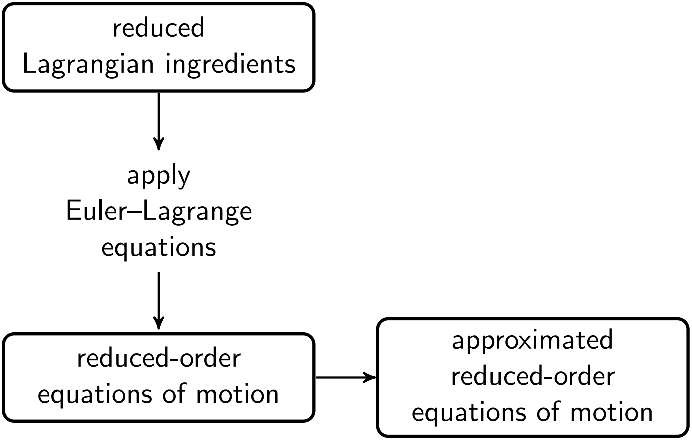
- interpolate functions before performing Galerkin projection

$$\Phi^T\tilde{f}_1(\ddot{q}_r;\mu) + \Phi^T\tilde{f}_2(\dot{q}_r;\mu) + \Phi^T\tilde{f}_3(q_r;\mu) = \Phi^T\tilde{f}^{\mathrm{ext}}(t;\mu)$$

$\tilde{f} = \Phi_f[Z\Phi_f]^+ Zf$: least-squares approximation of f

- destroyed properties:
  2. mass matrix not symmetric: does not define a metric
  3. stiffness matrix not symmetric: does not derive from a potential-energy function
  4. dissipation matrix not symmetric: does not derive from a dissipation function

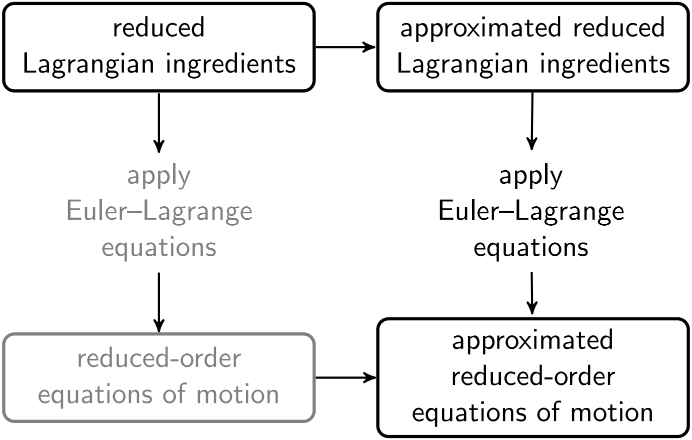
Existing complexity-reduction methods



+ leads to N-independent cost
- destroys Lagrangian structure

Outline

Proposed complexity-reduction method



+ leads to N-independent cost
+ preserves Lagrangian structure

Efficient, structure-preserving model reduction

- directly approximate reduced Lagrangian ingredients
  1. configuration space $Q_r = \mathbb{R}^m$ with $\mathbf{Q}_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
  2. Riemannian metric $\tilde{g}_r \approx g_r$
  3. potential-energy function $\tilde{V}_r \approx V_r$
  4. dissipation function $\tilde{\mathcal{F}}_r \approx \mathcal{F}_r$
  5. external force $\tilde{f}_r^{\mathrm{ext}} \approx f_r^{\mathrm{ext}}$

Approximated reduced Lagrangian ingredients

1. configuration space $Q_r = \mathbb{R}^m$ with $\mathbf{Q}_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
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External-force approximation $\tilde{f}_r^{\mathrm{ext}}$

- least-squares approximation of external force

$$\tilde{f}^{\mathrm{ext}} = \Phi_f \left[Z\Phi_f\right]^+ Zf^{\mathrm{ext}} \approx f^{\mathrm{ext}}$$

- apply Lagrange–D'Alembert principle to $\tilde{f}^{\mathrm{ext}}$ with variations in reduced configuration space:

$$\tilde{f}_r^{\mathrm{ext}} = \Phi^T \tilde{f}^{\mathrm{ext}} = \Phi^T \Phi_f \left[Z\Phi_f\right]^+ Zf^{\mathrm{ext}}$$

- Offline (expensive)
  1. collect snapshots of the external force and compute basis Φ_f
  2. determine sampling matrix Z
  3. compute small-scale matrix $A = \Phi^T \Phi_f \left[Z\Phi_f\right]^+$
- Online (cheap)
  1. compute a few entries of the external force Zf^{ext}
  2. compute small-scale product $A\left[Zf^{\mathrm{ext}}\right]$

Approximated reduced Lagrangian ingredients

1. configuration space $Q_r = \mathbb{R}^m$ with $\mathbf{Q}_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
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Riemannian-metric and dissipation-function approximations

$$g_r(v_r, w_r; \mu) = v_r^T \left[\Phi^T M(\mu)\Phi\right] w_r$$

$$\mathcal{F}_r(\dot{q}_r; \mu) = \dot{q}_r{}^T \left[\Phi^T C(\mu)\Phi\right] \dot{q}_r$$

- approximated quadratic ingredients:

$$\tilde{g}_r(v_r, w_r; \mu) = v_r^T \tilde{M}_r(\mu) w_r$$

$$\tilde{\mathcal{F}}_r(\dot{q}_r; \mu) = \dot{q}_r{}^T \tilde{C}_r(\mu) \dot{q}_r$$

- relies on approximating low-dimensional matrices

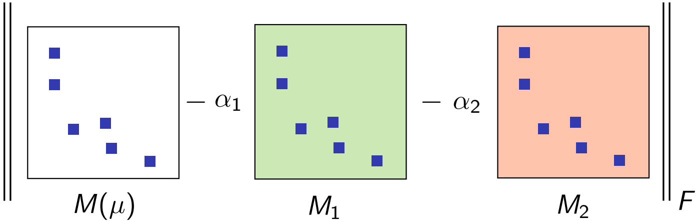
$$\tilde{M}_r(\mu) \approx \left[\Phi^T M(\mu)\Phi\right] > 0$$

$$\tilde{C}_r(\mu) \approx \left[\Phi^T C(\mu)\Phi\right] \geq 0$$

Mass-matrix approximation (similar for C)

- Offline (expensive)
  1. collect matrix snapshots $\{M_i\}$ and corresponding $\{\Phi^T M_i \Phi\}$
  2. determine 'sample entries'
- Online (cheap)
  1. compute only sample entries of $M(\mu)$
  2. solve cheap optimization problem for α_i:

$\underset{\alpha_1, \alpha_2}{\text{minimize}}$



subject to $\quad \alpha_1 \Phi^T M_1 \Phi + \alpha_2 \Phi^T M_2 \Phi > 0$

3. set $\tilde{M}_r(\mu) = \sum_i \alpha_i \Phi^T M_i \Phi$

Approximated reduced Lagrangian ingredients

1. configuration space $Q_r = \mathbb{R}^m$ with $\mathbf{Q}_r \equiv \{\Phi q_r \mid q_r \in Q_r\}$
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Potential-energy function approximation

$$V_r(q_r;\mu) \equiv V(\Phi q_r;\mu)$$

- replace Φ with a sparse matrix Ψ ($n_Z \ll N$ nonzero rows)

$$\tilde{V}_r(q_r;\mu) \equiv V(\Psi q_r;\mu).$$

- cost reduction
  - $\nabla_{q_r} V_r(q_r;\mu) = \Phi^T \nabla_q V(\Phi q_r;\mu)$ incurs $\mathcal{O}(Nm)$ flops
  - $\nabla_{q_r} \tilde{V}_r(q_r;\mu) = \Psi^T \nabla_q V(\Psi q_r;\mu)$ incurs $\mathcal{O}(n_Z m)$ flops
- compute Ψ by matching $\Psi^T \nabla_q V(\Psi q_r;\mu)$ and $\Phi^T \nabla_q V(\Phi q_r;\mu)$ for 'training' values of q_r and μ

Potential-energy function approximation

- Offline (expensive)
  1. collect snapshots of $\nabla_{q_r} V_r(q_r;\mu)$ for ‘training’ values of q_r, μ
  2. determine nonzero rows of Ψ
  3. solve optimization problem

$$\underset{\Psi}{\mathrm{minimize}} \sum_{j=1}^{J} \left\| \Psi^T \nabla_q V(\Psi {q_r}^j; \mu^j) - \Phi^T \nabla_q V(\Phi {q_r}^j; \mu^j) \right\|_2^2.$$

- Online (cheap): replace V_r with $\tilde{V}_r$

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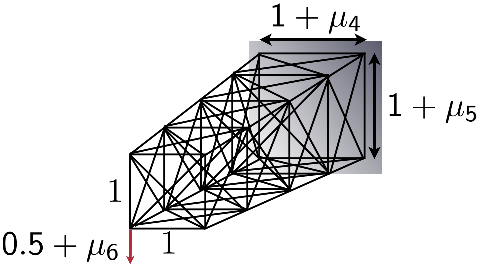
5. **Numerical example**

Simple example: conservative clamped-free truss



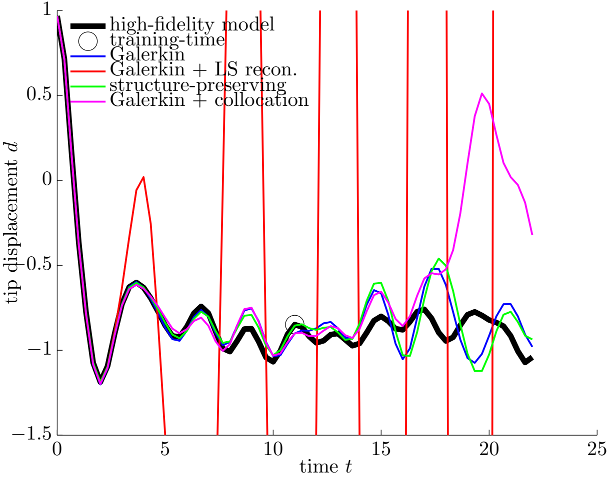
$$M(\mu)\ddot{q} + \nabla_q V(q;\mu) = 0$$

- V: potential energy, *high-order nonlinearity in q*
- density $\rho = 1 + \mu_1$
- bar cross-sectional area $A = 1 + \mu_2$
- modulus of elasticity $E = 1 + \mu_3$
- $\mu_i \in [-1, 1]$, $i = 1, \ldots, 6$
- 120 dofs in 'high-fidelity' model
- time integrator: implicit midpoint rule (symplectic)

Reduced-order models

1. Galerkin projection
   - + preserves structure
   - - expensive
2. Galerkin projection + collocation
   - - destroys structure
   - + cheap
3. Galerkin projection + gappy POD approximation of residual
   - - destroys structure
   - + cheap
4. proposed method
   - + preserves structure
   - + cheap

- reduced-order-model parameters
  - $\Phi \in \mathbb{R}^{N\times m}$: POD, $m = 18$ chosen via 99% 'energy criterion'
  - sample indices $n_Z = 30$
  - $\Phi_f \in \mathbb{R}^{N\times m_f}$: POD, $m_f = m = 10$
  - train at 3 configurations, test at a new configuration



	Galerkin	Galerkin + collocation	Galerkin + LS recon.	proposed method
error	6.85%	18.7%	690%	7.0%
speedup	0.41	1.77	2.06	1.82

Conclusions

- directly approximate reduced Lagrangian ingredients
  - + Lagrangian-structure preservation
  - + computational efficiency
- only reduced-order model delivering accuracy and speedup!
- future work
  - deploy on more realistic (larger, more highly nonlinear) problem
  - apply framework to preserve structure for other systems

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